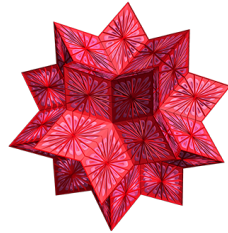Ordinary Differential Equations - Math 2280

Mathematica Commands: From Basics To ODE's - Version 14©

1. You may access the required software, called Mathematica, in the following three ways. 1. Get your own free copy by filling out the form at https://www.weber.edu/software/mathematica_request.html . 2. Any campus computer lab, including Tracy Hall Computer Lab, TY 126, and Elizabeth Hall Computer Lab, EH 213. 3. Virtual Lab: See http://www.weber.edu/virtuallab .

2. Here is how to get on Mathematica:

   Log in on a computer. Move the mouse pointer to the **Mathematica icon**, which looks like the above star like symbol, and rapidly **click twice** using the **left mouse button**. You may find the mathematica icon on the desktop, under the button Start - Programs - Wolfram or Mathematica, or by doing a search for Mathematica. Then click on NewDocument , or on New Notebook, if a notebook doesn't automatically open. If you need help, ask the lab aide.

3. Entering information is like using a word processor. You can also the arrow and backspace keys for editing. Try typing some text and practice editing it. Mathematica displays all works in "cells". The bracket on the far right indicate the cells. You can create a new cell for your next input by clicking the mouse below the cell you have been working in. First a horizontal line will appear and then when you type a new cell will be created.

   Throughout this lab and future labs input the statements in `True Type` font exactly as is!

4. Mathematica will do arithmetic operations using the usual symbols $+, -, *, /, \hat{} $. A blank space between two terms also indicates multiplication but using * might facilitate later editing. To see how Mathematica works, enter the following calculation followed by the appropriate Evaluation (Execution) key stroke; the $\boxed{\textbf{Enter}}$ key in the number pad (not the Return key $\boxed{\longleftarrow}$ in the main keyboard) should work for all computers.

   $$4\text{*}8 \quad \text{the } \boxed{\textbf{Enter}} \text{ key in the number pad (not the Return key } \boxed{\longleftarrow} \text{)}$$

   Note: Holding down $\boxed{\textbf{Shift}}$ key and then clicking on $\boxed{\longleftarrow}$ will also work in Windows machines.

   Now, evaluate the following. Check the answers using a calculator. If the output is the same as your input, you probably have made an error. Remember to push the $\boxed{\textbf{Enter}}$ key for Evaluation after each one.

   (a) `3+6*5` (Does Mathematica follow order of operation rules?)

   (b) `274/3` (Try this again with a decimal point after either number.)

   (c) `(3.24*6.791-14.7)/(4.5+82/3)`
       Mathematica will not accept brackets or braces as computational grouping symbols. Try `[(3.24*6.791)-14.7]/(4.5+82/3)`. Do you get the same answer?

   (d) `2.6^2.99`

   (e) `3 5` (Don't forget the space between 3 and 5.)

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5. All built-in Mathematica commands, functions, or symbols begin with a capital letter and require "brackets" to enclose the function argument. Type in and evaluate the followoing.

   (a) `Log[12.7]`   (This is the natural logarithm $\ln 12.7$.)

   (b) `E^2.5`   (This is $e^{2.5}$.)

   (c) `Log[E,12.7]`   (This is the same as `Log[12.7]` or $\ln 12.7$.)

   (d) `Log[10,72.8]`   (This is the common logarithm $\log 72.8$.) Notice that the first number in the argument indicates the base.

   (e) `Sqrt[64.0]`   (This is $\sqrt{64}$.)

   (f) `Abs[-2.5]`   (This is $|-2.5|$.)

   (g) `I*I`   I is $i = \sqrt{-1}$.

6. You can find a numerical approximation using the function `N` and `NumberForm`. Type in and evaluate each of the following.

   (a) `E`   (This symbol stands for the exact value of e and will only be replaced with its numerical value if asked.)

   (b) `N[E]`   (This gives the approximate value of e.)

   (c) `Pi`   (This symbol stands for the exact value of π and will only be replaced with its numerical value if asked.)

   (d) `N[Pi]`   (This gives the approximate value of π.)

   (e) `Pi//N`   (This is another ay of asking for the numerical value of π.)

The function `N` calculates the answer to 16 significant figures (not decimal places) but Mathematica only displays the default number of places. For viewing the answer to more than the default number of display places but **no more than 16** significant figures you must use the `NumberForm` commans along with `N`. Try the following.

   (f) `NumberForm[N[E], 10]`

   (g) `NumberForm[N[Pi], 16]`

You can use the function `N` itself for calculating and displaying the answer for **more than 17** significant figures. Try the following.

   (h) `N[Pi, 17]`   (Did you get one more digit than the last part?)

   (i) `N[Pi, 1000]`   (This gives π to 1000 significant figures!)

7. You can also input mathematics the same way you write it using palettes. A basic palette is usually present on the top right hand side of the screen. (if it is not, you can get it by clicking consecutively on the buttons Palettes, and Basic Math Assistant.) You can get a specific form by clicking on it in the palette. You can fill a form by clicking on each location and typing. You can also **Tab** between different locations in a form. Use the palette to type in and evaluate each of the following.

   (a) $3^{5.2}$

   (b) $\sqrt{37.4}$

   (c) $\sqrt{3.1^{2.7}}$

   (d) $\frac{125.3}{72}$

   (e) The numerical value of $\sqrt{\frac{5}{9}}$

8. The % key allows you to call back a previous output line. To call back the output from the previous line enter %, two lines up %%, 3 lines up %%%, etc. Evaluate the following.

   (a) `N[E^2]`

   (b) `Log[%]`   (This is the natural logarithm of the last output or $\ln(e^2)$ which, of course, is equal to 2.)

   (c) `Log[%%]`   (This is the natural logarithm of the two outputs before, which is again $\ln(e^2) = 2$.)

   (d) `Log[%]`   (What is this? Verify your answer!)

9. Mathematica can work symbolically to perform operations. Try these.

   (a) `Simplify[3x^2-x-9+x^2+7x+5]`

   (b) `Cancel[(x^2-2x-3)/(x^2-9)]`

   (c) `Factor[x^4 - 1]`

   (d) `Factor[a s + b a s]`   (Spaces are essential to indicate product.)

   (e) `Expand[(x+2)^3]`

   (f) `Apart[x/((x-2)(x^2+3))]`   (This performs partial fractions on the fraction $\frac{x}{(x-2)(x^2+3)}$.)

   (g) `Together[x/(x+5)-1/(x-4)]`

   Mathematica has three symbols for equality let's try each case in the next three problems.

10. Replacement. A single equal sign indicates replacements. Enter the following. `x = 5; y = 12; z = a + b;` (Notice that you can either enter each on a separate line or all on one line, separated by a semicolon. The final semicolon suppresses the output.) Now find the following.

   (a) `x y^2`   (Remember to insert the space to indicate a product.)

   (b) `Expand[z^x]`   (This is $(a + b)^5$.)

   (c) `Clear[x,y,z]`   This clears the stated values for x, y and z. If you don't clear the values, Mathematica will continue to use them.

11. In Mathematica, an equation is denoted by using two equal signs. For example, the equation `3x+2==x-5` represents the equation $3x + 2 = x - 5$. You can find exact solutions of many equations using the `Solve` command. Mathematica can solve equations of one or more variables. Should you want solutions for more than one variable, you must indicate the variables you want to solve for in a list. Lists are indicated by enclosing the entries is braces. Notice the syntax on each of the following and enter them exactly as they appear. Since x and y had numerical values previously these values must be cleared.

   (a) `Solve[3x-8==4]`   The solution is given as a replacement value for the variable ($\{x \rightarrow$answer$\}$). (If the output is an empty set, either this equation has no solution or you have forgotten to clear the value of x earlier. Check your answer.)

   (b) `Solve[a x-2==3a,x]`   (The x after the comma indicates that the equation should be solved for x and not a.)

   (c) `Solve[{x-2y==4,x-1==5y},{x,y}]`   (If you wish to solve 2 or more equations you enter them as a list.)

   If an equation cannot be solved exactly, then a numerical solution may work. Below are two such examples.

   (d) `NSolve[2x-3==x^2-3x-4]`

   (e) `NSolve[{x^2+y^2==16,y==x^2-2x+2},{x,y}]`   (The solutions include points of intersections of the circle $x^2 + y^2 = 16$ and the parabola $y = x^2 - 2x + 2$.)

   (f) `FindRoot[Cos[x]==x^3,{x,1}]`   This command applies the Newton's method to the equation $\cos x = x^3$ with the starting value $x_1 = 1$.

12. The equal sign for entering a function becomes `:=` (not just $=$). The independent variables must be followed by an underline. Do not use a capital letter as the first letter in naming functions since it might get mixed up with built-in Mathematica functions. Enter the following functions and perform the indicated steps.
   `f[x_]:=x^2 - 5; g[x_]:=Log[x]/x`

   (a) `f[3]`

   (b) `g[f[x]]`

   (c) `g[f[3]]`

   (d) Obtain a numerical approximation for part c; (`N[g[f[3]]]`).

   (e) `Solve[f[x]==4]`   (This solves the equation $x^2 - 5 = 4$.)

13. You can draw a function, relation, or equation using the `Plot` command. Each expression must be followed by a list which gives the variable(s) and the domain (minimum value to maximum value). Enter the following.

   (a) `Plot[x^2-3x+4,{x,-10,10}]`

      Change the domain from -5 to 5 and then again from -100 to 100. You do not need a symmetric domain, try from 0 to 20.

   (b) Mathematica can plot parametric equations, in two or three dimensions. Try

      `ParametricPlot[{ 2 Cos[t], 3 Sin[t]},{t,0,2 Pi}]` (This is the graph of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.)

   (c) You can graph a polar equation using the `PolarPlot` command.

      `PolarPlot[Sin[2t],{t,0,2Pi}]` This is the graph of $r = \sin(2\theta)$. (I used t in place of θ since it was easier to type in!)

   (d) To graph two or more functions on the same coordinate system, enter them as a list.

      `Plot[{Cos[x],x^3},{x,-2Pi,2Pi}]` (This is the graph of the two functions $y = \cos x$ and $y = x^3$ on the same coordinate system.)

      over the graph. The coordinates will appear on the screen.

   (e) You can read the coordinates of a point on a graph by first right clicking on the graph and choosing Get Coordinates (or, first, Drawing Tools) in the window that opens up.

      Now, Try to read the coordinates of the point of intersection of the two curves in the last part. Is the x value of the point of intersection approximately what you got from the `FindRoot` command in the earlier part of this lab, or about 0.86?

14. We can use Mathematica to evaluate one dimensional limits. The `Limit` command of Mathematica has been improved, but one must still verify that the limit exists graphically or numerically. Enter the following.

   (a) To investigate $\lim\limits_{x \to 0} \frac{|x|}{x}$, first graph this function near $x = 0$.

      `Plot[Abs[x]/x, {x,-1,1}]` From this graph it is clear that the left and right limits as $x \to 0$ are not equal, and so, the limit does not exist.

   (b) `Limit[Abs[x]/x,x->0, Direction->"FromBelow"]` (This is $\lim\limits_{x \to 0^-} \frac{|x|}{x}$.)

   (c) `Limit[Abs[x]/x,x->0, Direction->"FromAbove"]` (This is $\lim\limits_{x \to 0^+} \frac{|x|}{x}$.)

   (d) `Limit[Abs[x]/x,x->0]` Mathematica can not determine a limit value, and you know the reason is that the limit does not exist.

   (e) You can evaluate $\lim\limits_{x \to \infty} \frac{\ln x}{\sqrt{x}}$ by the l'Hospital's rule. Mathematica can also evaluate it.

      `Limit[Log[x]/Sqrt[x],x->Infinity]`

15. Mathematica can do symbolic differentiation.

   (a) `D[x^2Sin[x]-3x+1,x]`   (This gives the derivative of $y = x^2 \sin x - 3x + 1$ with respect to x.)

   (b) You can also use the prime notation.
      `f[x_]:=x^3-2x^2+5;f'[x]`   Is $f'(x) = 3x^2 - 4x$?
      `f''[x]`   (This gives the second derivative $f''(x)$.)
      `D[f[x],{x,3}]`   (This gives the third derivative $f'''(x)$.)

   (c) You can also perform implicit differentiation.
      `D[x^2 y[x]^2 + x Sin[y[x]]==1,x]`   This differentiates the equation $x^2y^2 + x \sin y = 1$ with respect to x. (`y[x]` indicates y is a function of x.)
      `Solve[%,y'[x]]`   Now, you have solved for y'.

16. Mathematica does symbolic integration. Mathematica evaluates definite integrals both numerically, and, also, by doing symbolic integration first. Input the following.

   (a) `Integrate [x^2+1,x]`   (Mathematica gives the antiderivative with $c = 0$.)

   (b) `Integrate [x^2+1,{x,-1,2}]`   (This evaluates $\int_{-1}^{2} (x^2 + 1)\, dx$.)

   (c) Numerical Integration. Often we do not know the antiderivative of a function but we can approximate its definite integral numerically.

      `NIntegrate[Exp[-x^2],{x,0,Infinity}]`   (This evaluates $\int_0^\infty e^{-x^2}\, dx$.)

   (d) However, you can hand evaluate some integrals more efficiently than Mathematica. The easy way to evaluate $\int 3x^2(2010 - x^3)^{1999}\, dx$ is by u-substitution. Mathematica expands its integrand before finding its antiderivative.

      `Integrate[3x^2 (2010-x^3)^1999, x]`

17. The `Series` command can be used to find the power series expansion of a function about a given point up to a given desired power. The `Sum` command can find the value of some sseries.

   (a) `Series[E^x, {x,0,6}]`   (This is the Maclaurin series of $f(x) = e^x$ with all terms up to the sixth power stated explicitly and `O[x]`^{7} stands for all higher order terms.)

   (b) By adding the command `Normal` a series is truncated to a polynomial.
      `Normal[Series[Cos[x], {x,0,8}]]`   (This is the 8th-degree Maclaurin polynomial of $f(x) = \cos x$.)

   (c) `Series[Log[x], {x,1,5}]`   (This is the Taylor series expansion of $f(x) = \ln x$ about $x = 1$ with all term up to the fifth power stated explicitly.)

   (d) Mathematica can find the sum of certain series.

      `Sum[1/n^2, {n,1,Infinity}]`   (This gives the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.)

18. The direction field of a differential equation of the form $\frac{dy}{dx} = f(x, y)$ is the plot of the vectors $\vec{v} = \langle 1, f(x, y)\rangle$ at each grid point (x, y) in the plane. The Mathematics command for such a plot is `VectorPlot`. Consider the differential equation $\frac{dy}{dx} = y - x^2$.

   (a) `VectorPlot[{1,y-x^2}, {x,-3,3}, {y,-4,4}]`   (This is the direction field of the given ODE for $-3 \le x \le 3$ and $-4 \le y \le 4$.)

   (b) You can change the number of vectors by specifying the number of vectors.
`VectorPlot[{1,y-x^2}, {x,-3,3}, {y,-4,4}, VectorPoints->20]`

   (c) Notice that Mathematica draws all vectors with the same length. It uses the vector colors to indicate the length of vectors. To preserve the length of vectors, use the following command.
`VectorPlot[{1,y-x^2},{x,-3,3},{y,-4,4},VectorScaling->Automatic,VectorPoints->20]`

   (d) Mathematica can display the vectors as part of the approximate trajectories of the solutions using the `StreamPlot` command.
`StreamPlot[{1,y-x^2}, {x,-3,3}, {y,-4,4}]`

19. Using the `DSolve` and `NDSolve`command, we can find solutions to many differential equations.

   (a) The syntax for the general solution of $y' = f(x, y)$ is `DSolve[y'[x]==`$f(x, y)$`, y[x], x]`. Consider the equation $\frac{dy}{dx} = y - x^2$.
`DSolve[y'[x]==y[x]-x^2, y[x], x]`   (`C[1]` in the answer is the constant c_1.)

   (b) The syntax for the solution of the IVP $y' = f(x, y)$, $y(x_0) = y_0$ is `DSolve[{y'[x]==`$f(x, y)$`,` `y[x_0]==y_0}, y[x], x]`. Consider the IVP $\frac{dy}{dx} = y - x^2$, $y(0) = 2$.
`DSolve[{y'[x]==y[x]-x^2,y[0]==2}, y[x], x]`

   (c) Sometimes the ODE can only be solved numerically or you may just want to graph the solution. In that case the command `NDSolve` is used and its output is graphed. The syntax for this command is `NDSolve[{y'[x]==`$f(x, y)$`, y[x_0]==y_0}, y[x], {x,` x `min,` x `max}]`.
`NDSolve[{y'[x]==y[x]-x^2,y[0]==2}, y[x], {x, -3,3}]`   (Now, Mathematics has the numerical solution, a collection of points on the solution, for $-3 \le x \le 3$.) To graph it, do the following.
`Plot[Evaluate[y[x] /. %], {x, -3,3}]`   (`/.` means replace all and `y[x] /.  %` means replace `y[x]` with the last output.)

   (d) We can graph the direction field and a solution together. The following is the direction field for $\frac{dy}{dx} = y - x^2$ and graph of the solution of the IVP for the same ODE with the initial condition $y(0) = 2$.
`graph1=VectorPlot[{1, y - x^2}, {x, -3, 3}, {y, -4, 4}]`
`NDSolve[{y'[x]==y[x]-x^2, y[0]==2}, y[x], {x, -3, 3}]`
`graph2=Plot[Evaluate[y[x] /. %], {x, -3, 3}]`
`Show[graph1, graph2]`

   (e) To combine a direction field with graphs of solutions of several IVP's with for the same ODE, use the `Table` command. The following is as in the last part except that initial conditions are $y(0) = n$ for $n = -3, -2, -1, 0, 1, 2, 3$.
`graph1=VectorPlot[{1, y - x^2}, {x, -3, 3}, {y, -4, 4}]`
`Table[NDSolve[{y'[x]==y[x]-x^2, y[0]==n}, y[x], {x, -3, 3}],{n,-3,3,1}]`
`graph3=Plot[Evaluate[y[x] /. %], {x, -3, 3}]`
`Show[graph1, graph3]`

20. It is essential, as a courtesy to future users, and continued problem-free usage, to leave the computers as you find them. When you are done, close the application and/or log off properly.